Higher Dimensional Homology Algebra II:Projectivity

Fang Huang, Shao-Han Chen, Wei Chen, Zhu-Jun Zheng*

Abstract: In this paper, we will prove that the 2-category (2-SGp) of symmetric 2-groups and 2-category ($\mathcal{R}$-2-Mod) of $\mathcal{R}$-2-modules([6]) have enough projective objects, respectively.

Keywords: Symmetric 2-Groups; Projective Objects; $\mathcal{R}$-2-Modules

1 Introduction

The 2-category ($\mathcal{R}$-2-Mod) of $\mathcal{R}$-2-modules should be important like the category (R-Mod) in classical homology algebra (we call it 1-dimensional homology algebra). The property of projective enough of the category (R-Mod) is a stone for constructing derived functor and derived category[4, 8, 11]. We believe that the property of projective enough of 2-category ($\mathcal{R}$-2-Mod) play the same role in higher dimensional homology algebra as the category (R-Mod) in 1-dimensional homology algebra.

In [1], D. Bourn and E.M. Vitale gave the definition of projective objects in the 2-category (2-SGp) of symmetric categorical groups(we call them symmetric 2-groups) and said that "another problem concerns projective objects (in the sense of Definition 11.1) in the 2-category of symmetric categorical groups. The notion of projectivity is crucial in the classical theory, but, unfortunately, we do not know if the 2-category of symmetric categorical groups has enough projective objects. (It would be interesting to solve this problem in order to appreciate the strong specialization done in Sections 14 and 15, where we consider only $\mathcal{F}$-extensions.)"

The main aim of this paper is try to prove the conjecture of D. Bourn and E. M. Vitale. In fact, we prove that the 2-categories (2-SGp) and ($\mathcal{R}$-2-Mod) have enough projective objects from the well-known result that the abelian category (R-Mod) has enough projective objects.

The present paper is organized as follows. In section 2, we will recall some basic facts

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on symmetric 2-groups and their extensions, which are appeared in [1, 2, 3, 7], and give the definition of projective objects in ($\mathcal{R}$-2-Mod)([6]). In the next two sections, we will proof (2-SGp) and ($\mathcal{R}$-2-Mod) have enough projective objects.

This is the second paper of the series works on higher dimensional homology algebra. The first paper is "2-Modules and the Representation of 2-Rings[5]". In the coming papers, we shall give the definition of injective object in the 2-category ($\mathcal{R}$-2-Mod), prove that this 2-category has enough injective objects and develop the (co)homology theory of it.

2 Preliminary

In this section, we will give the basic definitions and results cited from [1, 2, 3, 7].

Definition 1. [1, 3, 7] For a sequence $(\Gamma, \varphi, \Sigma)$ in (2-SGp) as in the following diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "0"] \arrow[dr, "\Gamma"'] & \arrow[d, phantom, "\Uparrow\varphi"] & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "\Sigma"'] & \end{tikzcd}$$

By the universal properties of kernel and cokernel([2, 3, 7]), there are homomorphisms Γ_0, Σ_0 as in the following diagram:

$$\begin{tikzcd} & Coker\Gamma \arrow[dr, "\Sigma_0"] & \\ \mathcal{A} \arrow[r, "\Gamma"] \arrow[dr, "\Gamma_0"'] & \mathcal{B} \arrow[u, "p_\Gamma"] \arrow[r, "\Sigma"'] & \mathcal{C} \\ & Ker\Sigma \arrow[u, "e_\Sigma"'] & \end{tikzcd}$$

The sequence $(\Gamma, \varphi, \Sigma)$ is 2-exact if it satisfies one of the following equivalent conditions:

1) $\Gamma_0 : \mathcal{A} \to Ker\Sigma$ is full and essentially surjective;

2) $\Sigma_0 : Coker\Gamma \to \mathcal{C}$ is full and faithful.

Remark 1. There are four equivalent conditions in above definition from Proposition 6.2 in [7].

Definition 2. [1] Let $\mathcal{A}, \mathcal{C}$ be in (2-SGp). An extension of $\mathcal{A}$ by $\mathcal{C}$ is a diagram $(\Gamma, \varphi, \Sigma)$ in (2-SGp)

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "0"] \arrow[dr, "\Gamma"'] & \arrow[d, phantom, "\Uparrow\varphi"] & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "\Sigma"'] & \end{tikzcd}$$

which satisfies the following equivalent conditions:

1) The triple $(\Gamma, \varphi, \Sigma)$ is 2-exact, Γ is faithful and Σ is essentially surjective;

2) Γ_0 is an equivalence and Σ is essentially surjective;

3) Γ is faithful and Σ_0 is an equivalence.

Next, we will only consider one special case in the definition of projective objects in (2-SGp) given by D. Bourn and E.M. Vitale in [1].

Definition 3. [1] Let $\mathcal{P}$ be a symmetric 2-groups. $\mathcal{P}$ is called projective if, for each 1-morphism $G : \mathcal{P} \to \mathcal{B}$, and each essentially surjective functor $F : \mathcal{A} \to \mathcal{B}$ in (2-SGp), there exist $G^{'} : \mathcal{P} \to \mathcal{A}$, and $g : F \circ G^{'} \Rightarrow G$ in (2-SGp).

Similar as the methods in (2-SGp), we have

Definition 4. An object $\mathcal{P}$ in ($\mathcal{R}$-2-Mod)([6]) is called a projective object, if for any $\mathcal{R}$-homomorphism $G : \mathcal{P} \to \mathcal{C}$, and any essentially surjective $\mathcal{R}$-homomorphism $F : \mathcal{B} \to \mathcal{C}$, there exist an $\mathcal{R}$-homomorphism $G^{'} : \mathcal{P} \to \mathcal{B}$, and 2-morphism $h : F \circ G^{'} \Rightarrow G$ in ($\mathcal{R}$-2-Mod).

3 Main Results I

In this section, we will show that (2-SGp) has enough projective objects from the basic results of 1-dimensional homological algebraic theory.

Notation[1, 2, 7]. For an abelian group G, we write G_{dis} for the symmetric 2-group with objects which are the elements of G, morphism of $a \to b$ is only the identity when $a = b$, the monoidal structure is induced from the group structure of G. Moreover, for a symmetric 2-group $\mathcal{G}$, we write $\pi_0(\mathcal{G})$ for the abelian group with the elements which are objects of $\mathcal{G}$ up to isomorphism(denote by $[b]$, for $b \in obj(\mathcal{G})$), equipped with monoidal structure $+$ of $\mathcal{B}$ as the operation and with the unit object 0 as the unit element.

Lemma 1. *Given a surjective group homomorphism $f : B \to C$ of abelian groups B and C. There is an essentially surjective morphism $F : B_{dis} \to C_{dis}$ of symmetric 2-groups.*

Proof. There is a functor

$$F : B_{dis} \longrightarrow C_{dis}$$
$$b \mapsto f(b),$$
$$b \xrightarrow{id} b \mapsto f(b) \xrightarrow{id} f(b)$$

Also, $F(b_1 + b_2) = f(b_1 + b_2) = f(b_1) + f(b_2) = F(b_1) + F(b_2)$. Then F is homomorphism of symmetric 2-groups.

Then for any $c \in obj(C_{dis})$, i.e $c \in C$. From the surjective group homomorphism f, there exists $b \in B$, such that $f(b) = c$. Then there exists an object b in B_{dis}, and identity morphism $F(b) = c$. So, F is essentially surjective. $\square$

Lemma 2. *Given an essentially surjective homomorphism $F : \mathcal{B} \to \mathcal{C}$ of symmetric 2-groups. There is a surjective group homomorphism $F_0 : \pi_0(\mathcal{B}) \to \pi_0(\mathcal{C})$.*

Proof. There is a group homomorphism

$$F_0 : \pi_0(\mathcal{B}) \longrightarrow \pi_0(\mathcal{C})$$
$$[b] \mapsto [F(b)]$$

which is well-defined, since if b and $b^{'}$ are in same equivalent class, i.e. there is an isomorphism $\alpha : b \to b^{'}$ in $\mathcal{B}$, and for F is a functor, so there is an isomorphism $F(\alpha) : F(b) \to F(b^{'})$, then $F(b)$ and $F(b^{'})$ are the same element in $\pi_0(\mathcal{C})$. Moreover, for any $[b_1], [b_2] \in \pi_0(\mathcal{B})$,

$$F_0([b_1] + [b_2]) = F_0([b_1 + b_2]) = [F(b_1 + b_2)],$$

$$F_0([b_1]) + F_0([b_2]) = [F(b_1)] + [F(b_2)] = [F(b_1) + F(b_2)],$$

there is an isomorphism $F_+(b_1, b_2) : F(b_1 + b_2) \to F(b_1) + F(b_2)$, such that $[F(b_1 + b_2)] = [F(b_1) + F(b_2)]$ in $\pi_0(\mathcal{C})$. Then F_0 is group homomorphism.

Then, for any $[c] \in \pi_0(\mathcal{C})$, choose a representative element $c \in obj(\mathcal{C})$ of $[c]$. For $c \in obj(\mathcal{C})$, and essentially surjective morphism F, there exist $b \in obj\mathcal{B}$ and an isomorphism $g : F(b) \to c$ in $\mathcal{C}$. Then, for $[c] \in \pi_0(\mathcal{C})$, there exists $[b] \in \pi_0(\mathcal{B})$, such that $F_0([b]) = [F(b)] = [c]$, i.e. F_0 is surjective. $\square$

Lemma 3. *Given a projective object P in (Ab), where (Ab) is the category of abelian groups([11, 12]). Then P_{dis} is a projective object in (2-SGp).*

Proof. For each essentially surjective morphism $F : \mathcal{A} \to \mathcal{B}$ and morphism $G : P_{dis} \to \mathcal{B}$ in (2-SGp), from Lemma 2, there are group homomorphisms $F_0 : \pi_0(\mathcal{A}) \to \pi_0(\mathcal{B})$ and $G_0 : \pi_0(P_{dis}) \to \pi_0(\mathcal{B})$, with F_0 is a surjection and $\pi_0(P_{dis}) = P$.

For P is projective object in (Ab), There exists $G_0^{'} : P \to \pi_0(\mathcal{A})$, such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[d, "G_0"] \arrow[dl, dashed, "G_0^{'}"'] \\ \pi_0(\mathcal{A}) \arrow[r, "F_0"'] & \pi_0(\mathcal{B}) \end{tikzcd}$$

From group homomorphism $G_0^{'} : P \to \pi_0(\mathcal{A})$, define a morphism

$$\begin{aligned} G^{'} : P_{dis} &\longrightarrow \mathcal{A} \\ x &\mapsto G^{'} \triangleq G_0^{'}(x), \\ x \xrightarrow{id} x &\mapsto G_0^{'}(x) \xrightarrow{id} G_0^{'}(x) \end{aligned}$$

where $G_0^{'}(x)$ is the representative element of the equivalent class $G_0^{'}(x)$ in $\pi_0(\mathcal{A})$, and $G^{'}(x_1 + x_2) = G_0^{'}(x_1 + x_2) = G_0^{'}(x_1) + G_0^{'}(x_2) = G^{'}(x_1) + G^{'}(x_2)$, for $x_1, x_2 \in obj(P_{dis})$.

Moreover, for $x \in obj(P_{dis}) = P$, there is $F_0(G_0^{'}(x)) = G_0(x)$, and under the definitions of F_0 and G_0, we have $[F(G^{'}(x))] = [(F \circ G^{'})(x)] = [G(x)]$ in $\pi_0(\mathcal{B})$, then there is an isomorphism $h_x : F \circ G^{'})(x) \to G(x)$ in $\mathcal{B}$. It is easy to check that there is a 2-morphism $h : F \circ G^{'} \Rightarrow G$ in (2-SGp) by h_x.

From above, we proved that P_{dis} is a projective object in (2-SGp). □

Lemma 4. *Given a projective object $\mathcal{P}$ in (2-SGp). Then $\pi_0(\mathcal{P})$ is a projective object in (Ab).*

Proof. For each surjective morphism $f : A \to B$ and 1-morphism $g : \pi_0(\mathcal{P}) \to B$ in (Ab). From Lemma 1, we have an essentially surjective morphism $F : A_{dis} \to B_{dis}$ and a 1-morphism $\tilde{G} : (\pi_0(\mathcal{P}))_{dis} \to B_{dis}$, and there is a composition $G : \mathcal{P} \to (\pi_0(\mathcal{P}))_{dis} \to B_{dis}$. There exist a 1-morphism $G^{'} : \mathcal{P} \to A_{dis}$ and a 2-morphism $h : F \circ G^{'} \Rightarrow G$ in the sense of $\mathcal{P}$ is projective object in (2-SGp).

Define a group homomorphism

$$\begin{aligned} g^{'} : \pi_0(\mathcal{P}) &\longrightarrow A \\ [x] &\mapsto g^{'}([x]) \triangleq G^{'}(x) \end{aligned}$$

which is well-defined, since if $[x] = [x^{'}]$ in $\pi_0(\mathcal{P})$, there is an isomorphism $\alpha : x \to x^{'}$ in $\mathcal{P}$, and $G^{'}$ is a fuctor, there is a morphism $G^{'}(\alpha) : G^{'}(x) \to G^{'}(x^{'})$ in A_{dis}, so $G^{'}(x)$ must be equal to $G^{'}(x^{'})$ in A, i.e. $g^{'}([x]) = g^{'}([x^{'}])$.

Moreover, from 2-morphism $h : F \circ G^{'} \Rightarrow G$, we have a morphism $h_x : F(G^{'}(x)) \to G(x)$ in $\mathcal{B}$. Thus, we have $g^{'} \circ f = g$. □

The next lemma appeared in [7] as a fact without proof, here we will give its proof.

Lemma 5. *For a symmetric 2-group $\mathcal{A}$, there is a full and essentially surjective 1-morphism $H : \mathcal{A} \to (\pi_0(\mathcal{A}))_{dis}$ in (2-SGp).*

Proof. There is a homomorphism of symmetric 2-groups

$$\begin{aligned} H : \mathcal{A} &\longrightarrow (\pi_0(\mathcal{A}))_{dis} \\ a &\mapsto [a], \\ a_1 \xrightarrow{\alpha} a_2 &\mapsto [a_1] \xrightarrow{id} [a_2] \end{aligned}$$

obviously, H is well-defined homomorphism of symmetric 2-groups.

H is full: for any pair of objects a_1, a_2 in $\mathcal{A}$, and identity morphism $id : H(a_1) \to H(a_2)$ in $(\pi_0(\mathcal{A}))_{dis}$, i.e. $[a_1] = [a_2]$ in $\pi_0(\mathcal{A})$, and from the definition of $\pi_0(\mathcal{A})$, there is an isomorphism $\alpha : a_1 \to a_2$ in $\mathcal{A}$, such that $H(\alpha) = id$.

H is essentially surjective: for any object $[a]$ in $(\pi_0(\mathcal{A}))_{dis}$, choose one representative object $a \in \mathcal{A}$ of $[a]$, s.t. $H(a) = [a]$. ☐

Abelian category (Ab) has enough projective objects as the category of $\mathbb{Z}$-modules, i.e. for any abelian group A, there is a surjective morphism $f : P \to A$, with P projective[11].

Theorem 1. *(2-SGp) has enough projective objects, i.e. for any symmetric 2-group in (2-SGp), there is an essentially surjective homomorphism $F : \mathcal{P} \to \mathcal{A}$, with $\mathcal{P}$ projective object in (2-SGp).*

Proof. For any symmetric 2-group $\mathcal{A}$, we have an abelian group $\pi_0(\mathcal{A})$. Thus, for $\pi_0(\mathcal{A}) \in obj(Ab)$, there is a surjective morphism $h : P \to \pi_0(\mathcal{A})$, with P projective in (Ab). From Lemma 3, we know that P_{dis} is a projective object in (2-SGp), together with the full and essentially surjective morphism $H : \mathcal{A} \to (\pi_0(\mathcal{A}))_{dis}$[7], and the 1-morphism $G : P_{dis} \to (\pi_0(\mathcal{A}))_{dis}$ from Lemma 1, there exist a 1-morphism $F : P_{dis} \to \mathcal{A}$, and 2-morphism $h : H \circ F \Rightarrow G$ as in the following diagram

$$\begin{tikzcd}
 & P_{dis} \arrow[ld, dashed, "F"'] \arrow[d, "G"] \\
\mathcal{A} \arrow[r, "H"'] \arrow[ur, phantom, "\Rightarrow h"] & (\pi_0(\mathcal{A}))_{dis}
\end{tikzcd}$$

Next, we will show that $F : P_{dis} \to \mathcal{A}$ is an essentially surjective morphism in (2-SGp).

In fact, for any $a \in obj(\mathcal{A})$, there is $H(a) \in obj((\pi_0(\mathcal{A}))_{dis})$, and since G is an essentially surjective morphism, there exist $x \in obj(P_{dis})$, and isomorphism $\beta : G(x) \to H(a)$ in $(\pi_0(\mathcal{A}))_{dis}$. Using 2-morphism $h : H \circ F \Rightarrow G : P_{dis} \to (\pi_0(\mathcal{A}))_{dis}$, there is a morphism $h_x : H(F(x)) \to G(x)$, then we get a composition morphism $\beta \circ h_x : H(F(x)) \to H(a)$ in $(\pi_0(\mathcal{A}))_{dis}$, and since H is full, there is a morphism $\alpha : F(x) \to a$ in $\mathcal{A}$, such that $H(\alpha) = \beta \circ h_x$.

Then for any $a \in obj(\mathcal{A})$, there exist $x \in obj(P_{dis})$ and an isomorphism $\alpha : F(x) \to a$ in $\mathcal{A}$.

Denote $\mathcal{P} \triangleq P_{dis}$, we have an essentially surjective morphism $F : \mathcal{P} \to \mathcal{A}$, with $\mathcal{P}$ projective object in (2-SGp). ☐

4 Main Results II

Lemma 6. *For a given 2-ring $\mathcal{R}$([5]), $\pi_0(\mathcal{R})$ is a ring.*

Proof. From the symmetric 2-group $\mathcal{R}$, we have an abelian group $\pi_0(\mathcal{R})$ which is given as in Lemma 2, together with a multiplication given by the multiplication of $\mathcal{R}$, i.e. for $[r_1], [r_2]$ in $\pi_0(\mathcal{R})$, $[r_1] \cdot [r_2] \triangleq [r_1 \cdot r_2]$ under the multiplicity of $\mathcal{R}$. Also, the multiplicity of $\pi_0(\mathcal{R})$ satisfies the following conditions, for all possible elements of $\pi_0(\mathcal{R})$:

1. $([r_1] \cdot [r_2]) \cdot [r_3] = [r_1 \cdot r_2] \cdot [r_3] = [(r_1 \cdot rr_2) \cdot r_3] = [r_1(r_2 \cdot r_3)] = [r_1] \cdot ([r_2] \cdot [r_3])$;

2. There exists $1 \in \pi_0(\mathcal{R})$, which is the unit object in $\mathcal{R}$, with $1 \cdot [r] = [1 \cdot r] = [r] = [r \cdot 1] = [r] \cdot 1$;

3. $[r] \cdot ([s_0] + [s_1]) = [r] \cdot [s_0 + s_1] = [r \cdot (s_0 + s_1)] = [r \cdot s_0 + r \cdot s_1] = [r \cdot s_0] + [r \cdot s_1] = [r] \cdot [s_0] + [r] \cdot [s_1]$.

So, $\pi_0(\mathcal{R})$ is a ring. □

Lemma 7. *For a ring R, there is a 2-ring R_{dis} associated with R.*

Sketch of proof. R_{dis} is a category consisting of:

· Objects are just the elements of R;

· Morphism from r_1 to r_2 is identity if $r_1 = r_2$, otherwise, empty.

R_{dis} is a discrete symmetric 2-group for R is an abelian group.

R_{dis} is a 2-ring from R is ring. We can define the 2-ring structure of R_{dis} the structure of R.

Lemma 8. *Given an $\mathcal{R}$-2-module $\mathcal{M}$, then $\pi_0(\mathcal{M})$ is an $\pi_0(\mathcal{R})$-module. Conversely, for an R-module M, then M_{dis} is an R_{dis}-2-module.*

Proof. First, $\pi_0(\mathcal{M})$ is an abelian group, together with a binary operator

$$\begin{aligned} \cdot : \pi_0(\mathcal{R}) \times \pi_0(\mathcal{M}) &\to \pi_0(\mathcal{M}) \\ ([r], [m]) &\mapsto [r \cdot m], \end{aligned}$$

where $r \cdot m$ is the operation of $\mathcal{R}$ on $\mathcal{M}$.

Moreover, $(\pi_0(\mathcal{M}), \cdot)$ satisfies:

1. $[r] \cdot ([m_1] + [m_2]) = [r] \cdot [m_1 + m_2] = [r \cdot (m_1 + m_2)] = [r \cdot m_1 + r \cdot m_2] = [r \cdot m_1] + [r \cdot m_2] = [r] \cdot [m_1] + [r] \cdot [m_2]$;

2. $([r_1] + [r_2]) \cdot [m] = [r_1 + r_2] \cdot [m] = [(r_1 + r_2) \cdot m] = [r_1 \cdot m + r_2 \cdot m] = [r_1 \cdot m] + [r_2 \cdot m] = [r_1] \cdot [m] + [r_2] \cdot [m]$;

3. $([r_1] \cdot [r_2]) \cdot [m] = [r_1 \cdot r_2] \cdot [m] = [(r_1 \cdot r_2) \cdot m] = [r_1 \cdot (r_2 \cdot m)] = [r_1] \cdot [r_2 \cdot m] = [r_1] \cdot ([r_2] \cdot m)$;

4. $1 \cdot [m] = [1 \cdot m] = [m]$.

So, $\pi_0(\mathcal{M})$ is an $\pi_0(\mathcal{R})$-2-module.

Conversely, for an R-module M, there is a symmetric 2-group M_{dis}. Moreover, there is a bifunctor $\cdot : R_{dis} \times M_{dis} \to M_{dis}$ gave by $(r, m) \mapsto r \cdot m$ under the operation of R on M and natural identities from the axioms of R-module M. After basic calculations, M_{dis} is an R_{dis}-2-module.

$\square$

Lemma 9. *Let $f : M \to N$ be a surjective R-homomorphism of R-modules. Then there is an essentially surjective R_{dis}-homomorphism $F : M_{dis} \to N_{dis}$.*

Proof. There ia a functor

$$\begin{aligned} &F : M_{dis} \to N_{dis} \\ &m \mapsto F(m) \triangleq f(m), \\ &m \xrightarrow{id} m \mapsto F(m) \xrightarrow{id} F(m) \end{aligned}$$

and $F(r \cdot m) \triangleq f(r \cdot m) = r \cdot f(m) = r \cdot F(m)$, then F is an R_{dis}-homomorphism.

For any $n \in N_{dis} = N$, since f is surjective, there exists $m \in M = obj(M_{dis})$, such that $f(m) = n$, i.e. $F(m) = n$. Then F is an essentially surjective R_{dis}-homomorphism. $\square$

Lemma 10. *Let $F : \mathcal{M} \to \mathcal{N}$ be an essentially surjective $\mathcal{R}$-homomorphism of $\mathcal{R}$-2-modules. Then there is a surjective $\pi_0(\mathcal{R})$-homomorphism $f : \pi_0(\mathcal{M}) \to \pi_0(\mathcal{N})$.*

Proof. There ia a $\pi_0(\mathcal{R})$-homomorphism

$$\begin{aligned} f : \pi_0(\mathcal{M}) &\to \pi_0(\mathcal{N}) \\ [m] &\mapsto f([m]) \triangleq [F(m)] \end{aligned}$$

which is well-defined, if $[m] = [m^{'}]$, i.e. there is an isomorphism $\alpha : m \to m^{'}$, then there is an isomorphism $F(\alpha) : F(m) \to F(m^{'})$ in $\mathcal{N}$, i.e. $f(m) = [F(m)] = [F(m^{'})] = f(m^{'})$.

For any $[n] \in \pi_0(\mathcal{N})$, choose a representative element $n \in obj(\mathcal{N})$ of $[n]$, and since F is essentially surjective, there exist $m \in obj(\mathcal{M})$, and $\beta : F(m) \to n$. Then there exists $[m] \in \pi_0(\mathcal{M})$, such that $f([m]) = [F(m)] = [n]$ in $\pi_0(\mathcal{N})$, i.e. f is surjective. $\square$

Lemma 11. *For a projective object P in (R-Mod), there is a projective object P_{dis} in (R_{dis}-2-Mod).*

Proof. For any essentially surjective R_{dis}-homomorphism $F : \mathcal{M} \to \mathcal{N}$ and R_{dis}-homomorphism $G : P_{dis} \to \mathcal{N}$. We have a surjective R-homomorphism $f : \pi_0(\mathcal{M}) \to \pi_0(\mathcal{N})$ and an R-homomorphism $g : P \to \pi_0(\mathcal{N})$, and $\pi_0(P_{dis}) = P$ ([7]).

Since P is a projective object, there exists $g^{'}: P \to \pi_0(\mathcal{M})$ such that $f \circ g^{'} = g$. Then we get an R_{dis}-homomorphism

$$\begin{aligned} G^{'}: P_{dis} &\to \mathcal{M} \\ x &\mapsto G^{'} \triangleq g^{'}(x) \end{aligned}$$

where $g^{'}(x)$ is the representative element of the isomorphism class of $g^{'}(x)$ in $\pi_0(\mathcal{M})$. And from $f(g^{'}(x)) = g(x)$, i.e. $[F(G^{'}(x))] = [G(x)]$, there exists an isomorphism $h_x : F(G^{'}(x)) \to G(x)$ in $\mathcal{N}$, so defines a 2-morphism $h : F \circ G^{'} \Rightarrow G$. □

Theorem 2. *($\mathcal{R}$-2-Mod) has enough projective objects, i.e. for any $\mathcal{M} \in obj$($\mathcal{R}$-2-Mod), there exists an essentially surjective $\mathcal{R}$-homomorphism $F : \mathcal{P} \to \mathcal{M}$ with $\mathcal{P}$ projective object in ($\mathcal{R}$-2-Mod).*

Proof. For $\mathcal{M}$, $\pi_0(\mathcal{M}) \in obj((\pi_0\mathcal{R})$-Mod), and (($\pi_0\mathcal{R}$)-Mod) has enough projective objects([11]), there exists a surjective morphism $g : P \to \pi_0(\mathcal{M})$ with P projective object in ($\pi_0\mathcal{R}$-Mod). From Lemma 9, we have an essentially surjective $G : P_{dis} \to (\pi_0(\mathcal{M}))_{dis}$, together with full and essentially surjective morphism $H : \mathcal{M} \to (\pi_0(\mathcal{M}))_{dis}$(similar as Lemma 5), and P_{dis} is a projective object, there exist $F : P_{dis} \to \mathcal{M}$, and 2-morphism $h : H \circ F \Rightarrow G$.

Next, we will check that F is essentially surjective. For any $m \in obj(\mathcal{M})$, $H(m) \in obj(\pi_0(\mathcal{M})_{dis})$, and G is essentially surjective, there exist $x \in obj(P_{dis}) = P$ and isomorphism $\alpha : G(x) \to H(m)$. From $h : H \circ F \Rightarrow G$, $h_x : H(F(x)) \to G(x)$, together with α, we have the composition morphism $H(F(x)) \to H(m)$. Moreover, H is full, there exists a morphism $F(x) \to m$ in $\mathcal{M}$. □

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Fang Huang, Shao-Han Chen, Wei Chen
Department of Mathematics
South China University of Technology
Guangzhou 510641, P. R. China

Zhu-Jun Zheng
Department of Mathematics
South China University of Technology
Guangzhou 510641, P. R. China
and
Institute of Mathematics
Henan University
Kaifeng 475001, P. R. China
E-mail: zhengzj@scut.edu.cn